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A fourth order Runge-Kutta Method for the Numerical Solution of first order Fuzzy Differential Equations

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Abstract: We propose fuzzy version of Runge – Kutta method of order four, for the solution of first order linear fuzzy differential equations without converting them to crisp form. The accuracy and efficiency of the proposed method is illustrated by solving a fuzzy initial value problem with trapezoidal fuzzy number.

Keywords: Fuzzy number, Trapezoidal fuzzy number, Fuzzy Differential Equations, Runge – Kutta method, higher order derivatives etc.

1. INTRODUCTION

Fuzzy Differential Equations (FDEs) model have wide range of applications in many branches of engineering and in the field of medicine. The concept of fuzzy derivative was defined by Chang S.L. and Zadeh L.A. in[5]. It was followed up by Dubosis.D and Prade[6].who used extension principle in their approach. The term “fuzzy differential Equation” was introduced in1987 by Kandel.A and Byatt.W.J They have been many suggestions for definition of fuzzy derivative to study“fuzzy differential Equation”.In the litreture, there are several approaches to study fuzzy differential equations.The first and most popular one is Hukuhara derivative made by Puri.M.L Ralesu.D.A [17].Here the solution of fuzzy differential equation becomes fuzzier as time goes on. This approach does not reproduce the rich and varied behaviour of ordinary differential equations. Bede.B and Gal.S.G[3,4] have introduced another concept of derivatives called the generalized Hukukara derivative.Under this interpretation ,the solution of a fuzzy differential equation becomes less fuzzier as time goes on. The strong generalized derivative is defined for a large class of fuzzy number valued function than Hukuhara derivative. This case a fuzzy differential equation is not unique. it has two solution locally. Recently some Mathematicians have studied fuzzy differential equations by Numerical methods. Numerical Solution of fuzzy differential equations has been introduced by M.Ma, M.Friedman, and Kandel.A in [15] through Euler method. Taylors method and Runge –Kutta methods have also been studied by authors in [1],[2].

After a preliminary section we study differentiability of the H-difference .Then we solve first order linear fuzzy differential equations by fourth order Runge –kutta method here we provide also some examples and compare this result with exact solution followed by complete error analysis.

2. PRELIMINARIES

In this section, some basic definition are reviewed [7,8]

Defintion2.1. A Fuzzy Number is a generalization of a regular, real number in the sense that it does not refer to one single value but rather to a connected set of possible values. Where each possible valuhas

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its own weight between 0 & 1. This weight is called the membership function.

1. A fuzzy number is a quantity. (i.e) is expressed as a fuzzy set defining a fuzzy interval in the real number R.
2. Convex fuzzy set
$u\left(tx+(1-t)y\right)\geq \min\left\{u(x),u(y)\right\}\forall t\in\left[0,1\right], x,y\in R$
3. Normalized fuzzy set
(i.e.) $\exists\ x_0 \in R \text{ with } u(x_0)=1$
4. Its membership function is piecewise continuous of bounded support.

We represent an arbitrary fuzzy number by an ordered pair of functions. $\left(\underline{u}(r),\overline{u}(r)\right), 0\leq r\leq 1$

That satisfies the following requirements.

1) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0,1],with respect to any r
2) $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0,1] with respect to any r
3) $\underline{u}(r)\leq\overline{u}(r), 0\leq r\leq 1$ then

The r-level set is $[u]_r=\{x/u(x)\geq r\}, 0\leq r\leq 1$ is a closed and bounded interval, denoted by $[u]_r=\left[\underline{u}(r)\leq\overline{u}(r)\right]$ and clearly, $[u]_0=\{x/u(x)>0\}$ is compact.

Definition 2.2: A trapezoidal fuzzy number u is defined by four real number of the trapezoidal is the interval $[a,d]$ and its vertices at $x=b, x=c$.
Trapezoidal fuzzy number will be written as $u=(a,b,c,d)$ is defined as follows

$$\begin{cases} \dfrac{x-a}{b-a} & a\leq x\leq b \\ 1 & b\leq x\leq c \\ \dfrac{x-d}{c-d} & c\leq x\leq d \end{cases}$$

we will have:

$(1)\, u>0 \text{ if } a>0$ $(2)\, u>0 \text{ if } b>0$
$(3)\, u>0 \text{ if } c>0$ $(4)\, u>0 \text{ if } d>0$

Definition2.2.1: A Trapezoidal fuzzy number $A=(a,b,c,d)$ is said to be zero trapezoidal fuzzy number if $a=0,b=0,c=0,d=0.$

Definition2.2.2: A Trapezoidal fuzzy number $A=(a,b,c,d)$ is said to be non - negative trapezoidal fuzzy number if $a\geq 0$

Definition2.2.3: A Trapezoidal fuzzy number $A=(a_1,b_1,c_1,d_1)$ and $B=(a_2,b_2,c_2,d_2)$ is said to be equal trapezoidal fuzzy number if $a_1=a_2, b_1=b_2, c_1=c_2, d_1=d_2$

Definition 2.3: Interval Number is a closed and bounded set of real numbers

$$[a,b]=\{x: a\leq x\leq b\forall a,b,x\in R\}$$

$$A=[a_1,a_2]\ \&\ B=[b_1,b_2]$$

$$A(+)B=[a_1+b_1, a_2+b_2]$$

$$A(-)B=[a_1-b_2, a_2-b_1]$$

$$KA=\begin{cases}[ka_1, ka_2] & \text{if } k\geq 0\\ [ka_2, ka_1] & \text{if } k<0\end{cases}$$

$A(.)B=[p,q]$ where

$$p=\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$$

$$q=\max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$$

$$A(:)B=\begin{cases}[a_1,a_2](.)\left[\dfrac{1}{b_1},\dfrac{1}{b_2}\right] & \text{if } 0\notin[b_1,b_2]\\ \text{empty} & \text{if } b_1=b_2=0\end{cases}$$

3. THE FUZZY DERIVATIVE:

Definition 3.1 (H-Difference): Let be $u,v\in R$. If there exists $w\in R$ such that $u=v\oplus w$, then w is called the H-Difference of u and v and is denoted by $u\Theta v$.

Definition 3.2: (Hukuhara Derivative) [17] Consider a fuzzy mapping $F:(a,b)\to R$ and $t_0\in(a,b)$.We say that F is differentiable at $t_0\in(a,b)$ if there exists an element $F'(t_0)\in R$ such that for all $h>0$ sufficiently small

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$\exists F(t_0+h)\Theta F(t_0), F(t_0)\Theta F(t_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0+}\frac{F(t_0+h)\Theta F(t_0)}{h} = \lim_{h\to 0-}\frac{F(t_0)\ \Theta F(t_0-h)}{h}$$

exist and are equal to $F'(t_0)$.

Note that this definition of the derivative is very restrictive; for instance in [3, 4] the authors showed that if $F(t)=c.g(t)$ where c is a fuzzy number and $g:[a,b]\to R^+$ is a function with $g'(t)<0$, then F is not differentiable. To avoid this difficulty, the authors of [3, 4] introduce a more general definition of the derivative for fuzzy mappings.

Definition 3.3: (Generalized Fuzzy Derivative) [15, 16] Let $F:(a,b)\to R$ and $t_0\in(a,b)$. we say that F is strongly generalized differentiable at t_0 if there exists as element $F'(t_0)\in R$ such that

(i) For $h>0$ sufficiently small $\exists F(t_0+h)\Theta F(t_0), F(t_0)\Theta F(t_0-h)$, and the limits satisfy

$$\lim_{h\to 0}\frac{F(t_0+h)\Theta f(t_0)}{h} = \lim_{h\to 0}\frac{F(t_0)\Theta\ F(t_0-h)}{h} = F'(t_0)$$

(ii) For $h>0$ sufficiently small $\exists F(t_0)\Theta F(t_0+h), F(t_0-h)\Theta F(t_0)$, and the limits satisfy

$$\lim_{h\to 0}\frac{F(t_0)\Theta F(t_0+h)}{(-h)} = \lim_{h\to 0}\frac{F(t_0-h)\Theta\ F(t_0)}{(-h)} = F'(t_0)$$

(iii) For $h>0$ sufficiently small $\exists F(t_0+h)\Theta F(t_0), F(t_0-h)\Theta F(t_0)$ and the limits satisfy

$$\lim_{h\to 0}\frac{F(t_0+h)\Theta f(t_0)}{h} = \lim_{h\to 0}\frac{F(t_0-h)\Theta\ F(t_0)}{(-h)} = F'(t_0)$$

(iv) For $h>0$ sufficiently small $\exists F(t_0+h)\Theta F(t_0), F(t_0-h)\Theta F(t_0)$ and the limits satisfy

$$\lim_{h\to 0}\frac{F(t_0)\Theta f(t_0+h)}{(-h)} = \lim_{h\to 0}\frac{F(t_0)\Theta\ F(t_0-h)}{h} = F'(t_0)$$

h and (-h) at denominators mean $\frac{1}{h}$ and - $\frac{1}{h}$ respectively.

Remark 3.3.1: A function that is strongly differentiable as in cases (i) and (ii) of definition 3.3, will be referred as (i) - differentiable or as (ii) - differentiable, respectively.

Lemma3.3.2: If $u(t)=(x(t),y(t),z(t),w(t))$ is a trapezoidal fuzzy number valued function, then (a) if u is (i) - differentiable (Hukuhara differentiable) then $u'=(x',y',z',w)$.(b) if u is (ii) –differentiable then $u'=(w',z',y',x)$.

4.1 Fourth order Runge- kutta method for solving fuzzy differential Equations

Let us consider the first order fuzzy ordinary differential equations of the form

$$\begin{cases} y'(y)\ f(t,y) \\ y(t_0)=y_0 \qquad (1)\end{cases}$$

Equation (2) and (3) are the exact and approximate solutions of Equations (1) respectively

$$[Y(t_n)]_r=[\underline{Y}(t_n;r),\overline{Y}(t_n;r)] \_\_\_\_\_ (2)\ and$$

$$[y(t_n)]_r=[\underline{y}(t_n;r),\overline{y}(t_n;r)] \_\_\_\_\_ \quad (3)$$

By using Fourth order Runge kutta method for approximate solution is calculated as follows

$$\underline{y}(t_{n+1};r)=\underline{y}(t_n;r)+\sum_{j=1}^{4}w_j k_{j,1}(t_n,y(t_n;r))$$

$$\overline{y}(t_{n+1};r)=\overline{y}(t_n;r)+\sum_{j=1}^{4}w_j k_{j,2}(t_n,y(t_n;r))$$

where the w_j s are constants Then $k_{j,1} and k_{j,2}$ for j =1, 2, 3, 4 are defined as follow

$$k_{1,1}(t_n,y(t_n;r))=\min\ h\left\{f(t_n,u)/u\in(\underline{y}(t_n;r)),\overline{y}(t_n;r)\right\}$$

$$k_{1,2}(t_n,y(t_n;r))=\max h\left\{f(t_n,u)/u\in(\underline{y}(t_n;r)),\overline{y}(t_n;r)\right\}$$

$$k_{2,1}(t_n,y(t_n;r))=\min\ h\left\{f(t_n+\frac{h}{2},u)/u\in(p_{1,1}(t_n,y(t_n;r),p_{1,2}(t_n,y(t_n;r),\right\}$$

$$k_{2,2}(t_n,y(t_n;r))=\max\ h\left\{f(t_n+\frac{h}{2},u)/u\in(p_{1,1}(t_n,y(t_n;r),p_{1,2}(t_n,y(t_n;r),\right\}$$

$$k_{3,1}(t_n,y(t_n;r))=\min\ h\left\{f(t_n+\frac{h}{2},u)/u\in(p_{2,1}(t_n,y(t_n;r),p_{2,2}(t_n,y(t_n;r),\right\}$$

$$k_{3,2}(t_n,y(t_n;r))=\max\ h\left\{f(t_n+\frac{h}{2},u)/u\in(p_{2,1}(t_n,y(t_n;r),p_{2,2}(t_n,y(t_n;r),\right\}$$

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$$k_{4,1}(t_n, y(t_n;r)) = \min\ h\left\{ f(t_n + \frac{h}{2}, u) / u \in (p_{3,1}(t_n, y(t_n;r), p_{3,2}(t_n, y(t_n;r), \right\}$$

$$k_{4,2}(t_n, y(t_n;r)) = \max h\left\{ f(t_n + \frac{h}{2}, u) / u \in (p_{3,1}(t_n, y(t_n;r), p_{3,2}(t_n, y(t_n;r), \right\}$$

where

$$p_{1,1}(t_n, y(t_n;r)) = \underline{y}(t_n;r) + \frac{h}{2} k_{1,1}(t_n, y(t_n;r)$$

$$p_{1,2}(t_n, y(t_n;r)) = \overline{y}(t_n;r) + \frac{h}{2} k_{1,2}(t_n, y(t_n;r)$$

$$p_{2,1}(t_n, y(t_n;r)) = \underline{y}(t_n;r) + \frac{h}{2} k_{2,1}(t_n, y(t_n;r)$$

$$p_{2,2}(t_n, y(t_n;r)) = \overline{y}(t_n;r) + \frac{h}{2} k_{2,2}(t_n, y(t_n;r)$$

$$p_{3,1}(t_n, y(t_n;r)) = \underline{y}(t_n;r) + \frac{h}{2} k_{3,1}(t_n, y(t_n;r)$$

$$p_{3,2}(t_n, y(t_n;r)) = \overline{y}(t_n;r) + \frac{h}{2} k_{3,2}(t_n, y(t_n;r)$$

Now, using the initial conditions x_0, y_0 and the fourth order Runge – kutta formula, we compute

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \frac{1}{6}(k_{1,1}(t_n, y) + 2k_{2,1}(t_n, y) + 2k_{3,1}(t_n, y) + k_{4,1}(t_n, y))$$

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \frac{1}{6}(k_{1,2}(t_n, y) + 2k_{2,2}(t_n, y) + 2k_{3,2}(t_n, y) + k_{4,2}(t_n, y))$$

The exact and approximate solutions at $t_n\ 0 \le n \le N$ *are denoted by*

$[Y(t_n)]_r = [\underline{Y}(t_n;r), \overline{Y}(t_n;r)]$ *and* $[y(t_n)]_r = [\underline{y}(t_n;r), \overline{y}(t_n;r)]$ respectively. The solution is calculated by grid points

$$a = t_0 \le t_1 \le t_2 \le \dots\dots\dots\dots \le t_n = b \text{ and } h = \frac{(b-a)}{N} = t_{n+1} - t_n$$

$$\underline{Y}(t_{n+1};r) = \underline{Y}(t_n;r) + \frac{1}{6} F[t_n, y(t_n;r)]$$

$$\overline{Y}(t_{n+1};r) = \overline{Y}(t_n;r) + \frac{1}{6} G[t_n, y(t_n;r)]$$

And

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \frac{1}{6} F[t_n, y(t_n;r)]$$

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \frac{1}{6} G[t_n, y(t_n;r)]$$

The following lemmas will be applied to show the convergences of theses approximate i.e.

$$\lim_{h\to 0} \underline{y}(t,r) = \underline{Y}(t,r)$$

$$\lim_{h\to 0} \overline{y}(t,r) = \overline{Y}(t,r)$$

Lemma4.2: Let the sequence of number $\{w_n\}_{n=0}^{N}$ satisfy $|w_{n+1}| \le A|w_n| + B, 0 \le n \le N-1$ for some given positive constants A and B (proof [5]) then

$$|w_n| \le A^n |w_0| + B\frac{A^n - 1}{A-1}, 0 \le n \le N-1$$

The proof of Lemma (4.2) follows Lemma 2 of Ming Ma, Kandel.A (1999).

Lemma 4.3: Let the sequence of numbers $\{w_n\}_{n=0}^{N}$ $\{v_n\}_{n=0}^{N}$ satisfy

$$|w_{n+1}| \le |w_n| + A\max\{|w_n|, |v_n|\} + B$$

$|v_{n+1}| \le |v_n| + A.\max\{|w_n|, |v_n|\} + B$,for some given positive constants A and B, and denote

$$|u_n| = |w_n| + |v_n|, 0 \le n \le N$$

Then $|u_n| \le \overline{A}^n |u_0| + \overline{B}\dfrac{\overline{A}^n - 1}{\overline{A} - 1}, 0 \le n \le N,$

where $\overline{A} = 1 + 2A$ *and* $\overline{B} = 2B$

The proof of Lemma (4.3) follows Lemma 2 of Ming Ma, Kandel.A (1999).

Theorem 4.4: Let $F[t,u,v]$, $G[t,u,v]$ belongs to $C^4(k)$ and let the partial derivatives of F, G be bounded over R Then for arbitrary fixed $r: 0 \le r \le 1$, the approximate solutions converge uniformly in t to the exact solutions. This theorem is simply proved (see proof theorem 4.1 in [1])

5.NUMERICAL EXAMPLE

5.1 Numerical Example

Consider the Fuzzy initial value problem,

$$\begin{cases} y'(t) = y(t), t \in [0,1] \\ y(0) = (0.8 + 0.125r, 1.1 - 0.1r), 0 < r \le 1 \end{cases}$$

$$\underline{Y}(t;r) = \underline{y}(t;r)e^t,$$

$$\overline{Y}(t;r) = \overline{y}(t;r)e^t, \text{ which}$$

$$t = 1, Y(1;r) = [(0.8 + 0.125r)e, (1.1 - 0.1r)e]$$

Comparison between the exact solution and approximate solution of Fourth order Runge- kutta method for t =1 by complete error analysis.

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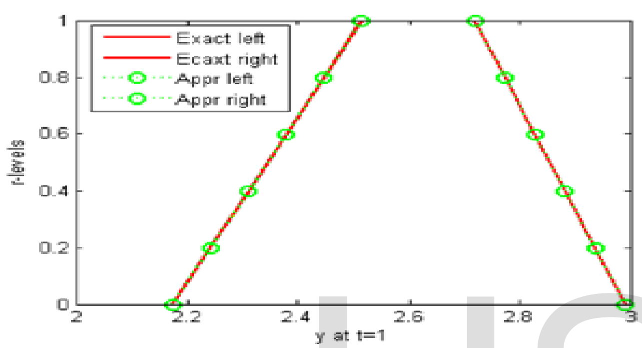
r	t	Exact solutions at t =1		Approximated solutions at h=0.01 RK4	
		$\underline{Y}$	$\overline{Y}$	$\underline{y}$	$\overline{y}$
0	1	2.174625	2.990110	2.174625	2.990110
0.2	1	2.242583	2.935744	2.242583	2.935744
0.4	1	2.310540	2.881379	2.310540	2.881379
0.6	1	2.378497	2.827013	2.378497	2.827013
0.8	1	2.446454	2.772647	2.446454	2.772647
1	1	2.514411	2.718282	2.514411	2.718282



6.CONCLUSIONS:

In this work, we have used the proposed fourth order Runge- Kutta method to find a numerical solution of fuzzy differential equations using trapezoidal fuzzy number. Taking into account the convergence order O (h^4) is obtained by the proposed method. Form this we see that our proposed fourth order Runge- Kutta method gives better solution than second order Runge- Kutta method which was
studied in [18]

REFERENCES

[1]. Abbasbandy.S and Allahviranloo.T, Numerical solution of fuzzy differential by Runge – Kutta method, J.Sci.Teacher Training University, 1(3), 2002.

[2]. Abbasbandy.S, Allahviranloo.T, Numerical Solution of fuzzy differential equations by Taylor method, J. of Computational methods in Applied Mathematics 2 (2002),113-124.

[3]. Bede.B. and Gal, S.G Generalizations of the differentiability of fuzzy number valued functions with applications to fuzzy differential equation, Fuzzy Sets and Systems 151, 581 – 599, 2005.

[4]. Bede. B., Rudas, I.J. and Bencsik, A.L. First order linear fuzzy differential equations under generalized differentiability, Inform.Sci.177, 1648-1662, 2007.

[5]. Chang.S.L, .Zadeh.L.A, On Fuzzy mapping and control, IEEETrans, Systems Man cybernet.2 (1972), 30-34.

[6]. Dubosis.D, Prade.H, towards fuzzy differential calculus: Part3, differentiation, Fuzzy sets and systems 8(1982), 25-233.

[7]. Ganesan.K and Veeramani.P (2006) fuzzy linear programming with trapezoidal fuzzy numbers Ann.oper.Res, 143,305-315

[8]. Goetschel.R, Voxman.Elementry.W Fuzzy calculus, Fuzzy Sets and Systems 18 (1986).31-43.

[9]. Kaleva.O, Fuzzy differential equations, Fuzzy sets and systems 24(1987).301-317

[10]. .Kaleva.O, the Cauchy problem for Fuzzy differential equations, Fuzzy sets and systems 35(1990), 389-396.

[11]. Kandel.A,Byatt.w.j,fuzzy differential equations in :proceedings of International Conference on Cybernetics and Society ,Tokyo (1978),pp.1213-1216

[12]. Kandel.A Byatt.W.J,fuzzy process,Fuzzy Sets and Systems,4(1980) 117-152

[13]. .Kaleva.O, Fuzzy differential equations, Fuzzy sets and systems 24(1987).301-317.

[14]. Kaleva.O, the Cauchy problem for Fuzzy differential equations, Fuzzy sets and System35(19990),389-396

[15]. Ming Ma, Friedman.M, and Kandel.A, Numerical solutions of fuzzy differential equations, Fuzzy Sets and Systems, 105 (1999), 35-48.

[16]. Ming Ma, M.Friedman, A.Kandel, Numerical solutions of fuzzy differential equations, Fuzzy sets and System 105(1999), 133-138.

[17]. Puri.M.L, Ralescu.D.A, Differentials of Fuzzy functions, J.Math.Anal.Appl.91 (1983) 552-558.

[18]. Parimala.V, Rajarajeswari .P, Nirmala. V, A second order Runge –kutta method to solve fuzzy differential equations with fuzzy initial condition 3(2014), 428-431